Special Bounds for Theta Sums

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October 12, 2024
PRIMES October Conference

Theta Sums

Definition 1

A **Jacobi theta function** is a sum of the form

$$\theta(\tau) = \sum_{n\in\mathbb{Z}} \mathrm{e}^{\pi \mathrm{i} n^2 \tau} = \sum_{n\in\mathbb{Z}} e\left(\tfrac{1}{2} n^2 \tau\right),$$

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A **modified theta function** is a sum of the form

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These are typical examples of a wider class of functions called **theta sums**.

Visualizing Theta Sums

We can evaluate theta sums along curves in the upper half plane. The curves we will look at are of the form

$$\{x + i\varepsilon \colon x \in \mathbb{R}, \varepsilon > 0, \varepsilon \text{ small}\}$$

Then we plot the output of the theta sum as we move along a small part of the curve by varying x in a small interval. In the following pictures, we plot the "normalized" function $\varepsilon^{1/4}\theta(x + i\varepsilon; \alpha, \beta)$.

Pictures of Bounded Theta Sums

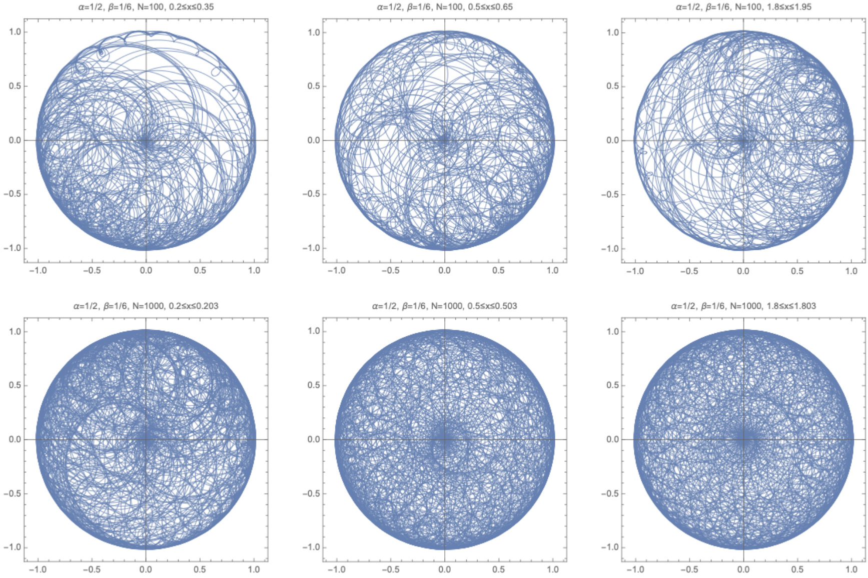
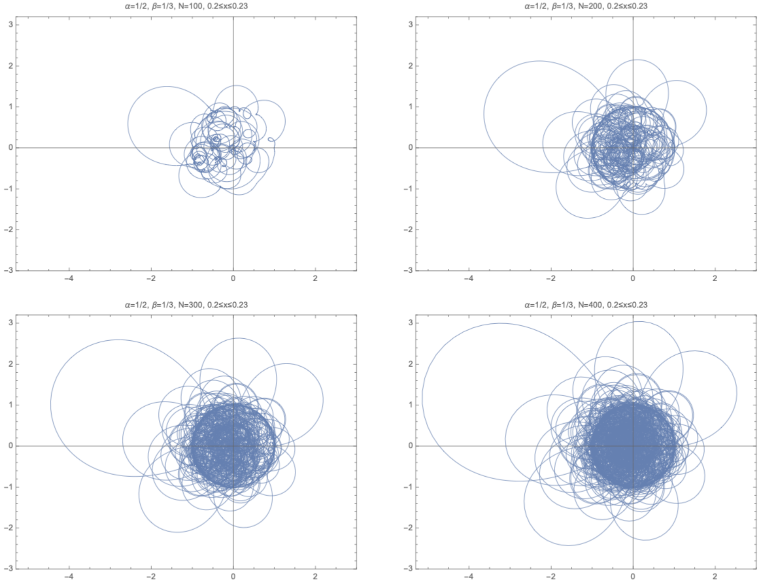


Image taken from [Cellarosi and Osman, 2023a]

Pictures of Unbounded Theta Sums



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Theta Sum Bounds

Theorem 3 ([Cellarosi and Osman, 2023a])

Let $\alpha = \frac{a}{2m}$ *and* $\beta = \frac{b}{2m}$*, with* a, b*, and* m *all odd, and such that* $\gcd(a, b, m) = 1$*. Then there exists a constant* $C = C(a, b, m)$ *such that*

$$|\theta(x + i\varepsilon; \alpha, \beta)| \leq C\varepsilon^{-\frac{1}{4}}$$

for every $x + \mathrm{i}\varepsilon$ *in the upper half plane.*

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for every $x + \mathrm{i}\varepsilon$ *in the upper half plane.*

- This theorem identifies which α and β produce a bound for our theta sum.
- If α and β are not of this form, then for any C, there exists $x + \mathrm{i}\varepsilon$ such that $|\theta(x + \mathrm{i}\varepsilon; \alpha, \beta)| > C\varepsilon^{-1/4}$.

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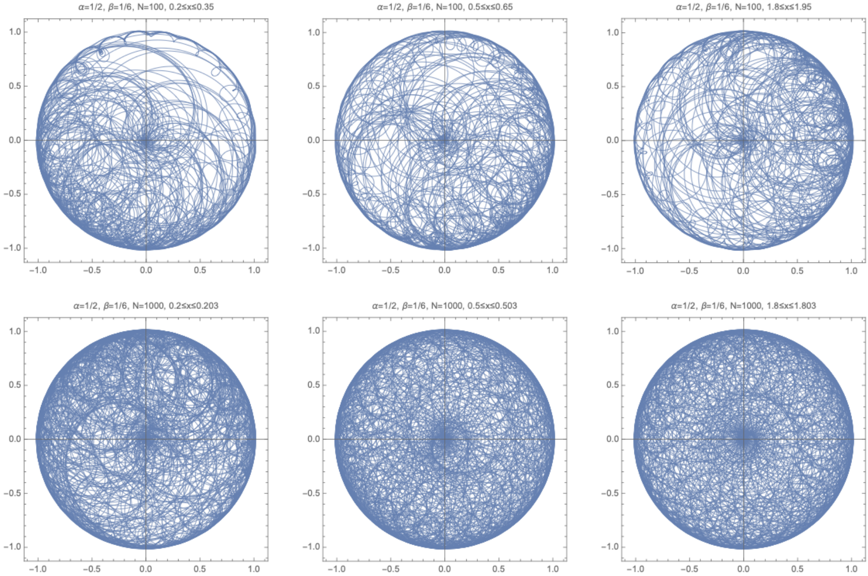

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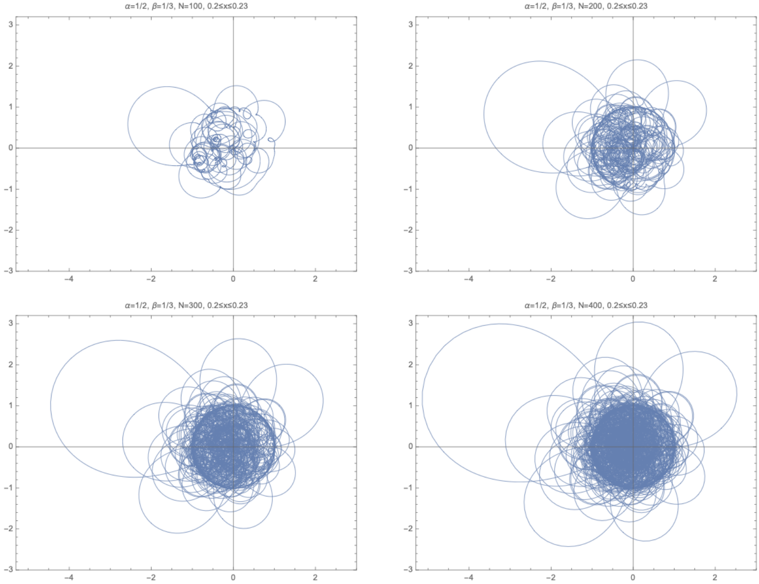

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Quadratic Forms

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A **quadratic form** is a polynomial expression where all terms have degree two.

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Some examples of quadratic forms:

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- $2a^2 + 3ab + b^2$
- $\sqrt{2}uv + u^2 + \sqrt{5}v^2 - \pi vw$

Rank 2 Theta Sum

Recall the Jacobi theta function:

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Definition 5

Let Q be a quadratic form in two variables. We define the **Siegel theta series** as

$$\Theta(\tau; \boldsymbol{\alpha}, \boldsymbol{\beta}) = \sum_{\boldsymbol{n}\in\mathbb{Z}^2} e\left(\tfrac{1}{2} Q(\boldsymbol{n} - \boldsymbol{\beta})\tau + (\boldsymbol{n} - \boldsymbol{\beta}) \cdot \boldsymbol{\alpha}\right).$$

Now $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are vectors in $\mathbb{R}^2$ instead of being real numbers.

Rank 2 Results

Theorem 6 (L., Osman)

Let $Q(u,v)$ be a quadratic form with no uv term. Let $\boldsymbol{\alpha} = \left(\frac{1}{2}, \frac{1}{2}\right)$ and $\boldsymbol{\beta} = \left(\frac{1}{2}, \frac{1}{2}\right)$. Then there exists a constant $C = C(\boldsymbol{\alpha}, \boldsymbol{\beta})$ such that

$$|\Theta(x + \mathrm{i}\varepsilon; \boldsymbol{\alpha}, \boldsymbol{\beta})| \leq C\varepsilon^{-\frac{1}{2}}$$

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Theorem 7 (L., Osman)

Let $Q(u, v)$ be a quadratic form whose coefficients are linearly independent over the rationals. Then for every $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ and for every $C > 0$, there exists an $x + \mathrm{i}\varepsilon$ such that

$$|\Theta(x + \mathrm{i}\varepsilon; \boldsymbol{\alpha}, \boldsymbol{\beta})| > C\varepsilon^{-\frac{1}{2}}.$$

Acknowledgments

- I would like to thank my mentor Dr. Tariq Osman for working together with me about theta sums and answering all of my questions about them.
- I would also like to thank the supervisor of the project, Prof. Dmitry Kleinbock, who helped me understand how PRIMES works.
- I also wish to thank Dr. Tanya Khovanova, the head mentor of PRIMES and PRIMES-USA for junior students, who sent many helpful resources for mathematical writing to me and other PRIMES students.
- Finally, I would like to thank the organizers of the MIT-PRIMES program for giving me the opportunity to conduct mathematical research.

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